

A REMAINDER FORMULA AND LIMITS OF CARDINAL SPLINE INTERPOLANTS

BY

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ABSTRACT. A Peano-type remainder formula

$$f(x) - S_n(f; x) = \int_{-\infty}^{\infty} K_n(x, t) f^{(n+1)}(t) dt$$

for a class of symmetric cardinal interpolation problems C.I.P. $(E, F, \mathbf{x})$ is obtained, from which we deduce the estimate $\|f - S_{n,r}(f; \cdot)\|_{\infty} \leq K \|f^{(n+1)}\|_{\infty}$. It is found that the best constant K is obtained when $\mathbf{x}$ comprises the zeros of the Euler-Chebyshev spline function. The remainder formula is also used to study the convergence of spline interpolants for a class of entire functions of exponential type and a class of almost periodic functions.

1. Introduction. As in [5], for $\mathbf{x} = (x_0, x_1, \dots, x_m)$, $0 = x_0 < x_1 < \dots < x_m = 1$ and incidence matrices $E = \|E_{ij}\|_{i=0}^m_{j=0}^n$, $F = \|F_{ij}\|_{i=0}^m_{j=0}^n$ with $E_{0j} = E_{mj}$ and $F_{0j} = F_{mj}$, $j = 0, \dots, n$, let $\mathcal{C}(F, \mathbf{x}) := \{f: \mathbf{R} \rightarrow \mathbf{C}; \forall \nu \in \mathbf{Z}, f|(\nu + x_i, \nu + x_{i+1}) \in \pi_n, i = 0, \dots, m-1, \text{ and } f^{(n-j)}(\nu + x_i^-) = f^{(n-j)}(\nu + x_i^+) \forall (i, j) \text{ with } F_{ij} = 0\}$, and refer to the following 'cardinal' interpolation problem as the C.I.P. $(E, F, \mathbf{x})$:

For sequences of numbers $\{y^{(i,j)}\} = \{y_{\nu}^{(i,j)}; 0 \leq i < m \text{ and } E_{ij} = 1\}$, find $S \in \mathcal{C}(F, \mathbf{x})$ satisfying $S^{(j)}(\nu + x_i) = y_{\nu}^{(i,j)}$.

Sufficient conditions for C.I.P. $(E, F, \mathbf{x})$ to be poised, i.e. existence of a unique $S \in \mathcal{C}(F, \mathbf{x})$, $S(x) = O(|x|^{\gamma})$ as $x \rightarrow \pm \infty$ satisfying $S^{(j)}(\nu + x_i) = y_{\nu}^{(i,j)}$ when $y_{\nu}^{(i,j)} = O(|\nu|^{\gamma})$, are given in [5].

Suppose that the C.I.P. $(E, F, \mathbf{x})$ is poised; then given a sufficiently smooth function f of power growth $\exists$ a unique $S_n(f; \cdot) \in \mathcal{C}(F, \mathbf{x})$ of power growth which interpolates f in the sense that

$$(1.1) \quad S_n^{(j)}(f; \nu + x_i) = f^{(j)}(\nu + x_i), \quad \nu \in \mathbf{Z}, E_{ij} = 1.$$

The following problem then arises.

Problem. Find necessary and sufficient conditions so that $S_n(f; \cdot) \rightarrow f$ uniformly as $n \rightarrow \infty$.

This question was first raised by Schoenberg [9] who also found a sufficient condition for the convergence of $S_n(f; \cdot)$ for the case where $m = 1$, n is odd, $E = F$ and

$$(1.2) \quad E_{0j} = E_{1j} = \begin{cases} 0, & j = 1, 2, \dots, n, \\ 1, & j = 0. \end{cases}$$

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In this case $\mathcal{C}(F, \mathbf{x})$ comprises odd degree cardinal splines with integer knots. Schoenberg [9] proves the following

THEOREM A. *Let*

$$(1.3) \quad f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iux} d\alpha(u)$$

where $\alpha(u)$ is a function of bounded variation in $[-\pi, \pi]$ and let $A = \alpha(-\pi + 0) - \alpha(-\pi)$, $B = \alpha(\pi) - \alpha(\pi - 0)$. Then

$$\lim_{k \rightarrow \infty} S_{2k-1}(f; x) = f(x) + \frac{i(A - B)}{2\pi} \sin \pi x$$

uniformly on $\mathbf{R}$.

A partial converse of Theorem A was given by Richards and Schoenberg [8]. Subsequently, Marsden and Riemenschneider [6] generalised Theorem A to the case of cardinal Hermite interpolation which corresponds to $m = 1$, $E = F$ and, for some $1 \leq r \leq \frac{1}{2}(n + 1)$,

$$E_{0j} = E_{1j} = \begin{cases} 1, & j = 0, 1, \dots, r - 1, \\ 0, & j = r, \dots, n. \end{cases}$$

In this case, a sufficient condition for convergence of $S_n(f; \cdot)$ analogous to that of Richards and Schoenberg was given by Goodman [4].

Recently, in an attempt to obtain more information on the convergence problem, I. J. Schoenberg [10] obtained a Peano type remainder formula

$$(1.4) \quad f(x) - S_{2k-1}(f; x) = \int_{-\infty}^{\infty} K_{2k-1}(x; t) f^{(2k)}(t) dt \quad (x \in \mathbf{R})$$

for the case $m = 1$, $n = 2k - 1$, $E = F$ and E satisfies (1.2), where $K_{2k-1}(x, t) := (x - t)_+^{2k-1} - S_{2k-1}((\cdot - t)_+^{2k-1}; x)$. From (1.4), Schoenberg [10] deduced Theorem A and also obtained a convergence result for a class of almost periodic functions.

In this paper we shall consider the symmetric C.I.P. $(E, F, \mathbf{x})$

$$(1.5) \quad \begin{aligned} x_i &= 1 - x_{m-i}, & i &= 0, \dots, m, \\ F_{ij} &= 1 & \text{iff } i &= 0 \text{ or } m \text{ and } j = 0, \dots, r - 1, \end{aligned}$$

for some $1 \leq r \leq n + 1$, and either

- (a) $n + r$ is even, $m = r$ and $E_{ij} = 1$ iff $j = 0$ and $i = 0, \dots, m$, or
- (b) $n + r$ is odd, $m = r + 1$ and $E_{ij} = 1$ iff $j = 0$ and $i = 1, \dots, m - 1$.

That this problem is poised follows from Corollary 4.3 of [5] and was earlier shown by Micchelli [7]. In this case the class of cardinal spline functions $\mathcal{C}(F, \mathbf{x})$ is usually denoted by $\mathfrak{S}_{n,r}$, and clearly

$$\mathfrak{S}_{n,r} \equiv \mathcal{C}(F, \mathbf{x}) = \{S \in C^{n-r}(\mathbf{R}); S|_{(\nu, \nu+1)} \in \pi_n, \forall \nu \in \mathbf{Z}\}.$$

For r, n as above, we define $\mathfrak{G}_{n,r} := \{f \in C^{n+1-r}(\mathbf{R}); f|_{(\nu, \nu+1)} \in C^n[(\nu, \nu + 1)] \text{ and } f^{(n)} \text{ bounded and absolutely continuous on } (\nu, \nu + 1), \forall \nu \in \mathbf{Z}\}.$

For $f \in \mathfrak{F}_{n,r}$ of power growth, we let $S_{n,r}(f; \cdot)$ denote the unique function of power growth that interpolates f as in (1.1). Following the approach of Schoenberg [10] we derive a formula for the remainder $f - S_{n,r}(f; \cdot)$ and deduce the following result.

THEOREM 1. *For fixed n, r and $\mathbf{x}$, $\exists K$ such that for any $f \in \mathfrak{F}_{n,r}$ with $f^{(n+1-r)}$ of power growth and $\|f^{(n+1)}\|_\infty < \infty$,*

$$(1.6) \quad \|f - S_{n,r}(f; \cdot)\|_\infty \leq K \|f^{(n+1)}\|_\infty,$$

and equality is attained for some $f \in \mathfrak{F}_{n,r}$. For fixed n, r , K is a minimum when $\mathbf{x}$ comprises the zeros of the Euler-Chebyshev spline $\mathfrak{E}_{n+1,r}$ (see [1] and [4]) and in this case equality is attained for $f = \mathfrak{E}_{n+1,r}$.

For $r = n + 1$, this result reduces to a classical result on optimal constants in the remainder for Lagrange interpolation by polynomials (see [3, p. 64]).

Now let $B_\sigma = \{f; f \text{ is the restriction to } \mathbf{R} \text{ of an entire function of exponential type } \leq \sigma \text{ and } \|f\|_\infty < \infty\}$.

By deriving bounds on the best constants K in (1.6), we prove the following results, all of which are generalisations of results of Schoenberg [10].

THEOREM 2. *For fixed $\mathbf{x}$ and $r \geq 1$, $\exists K_r$ such that for all $n \geq r - 1$ and $f \in B_\sigma$,*

$$\|f - S_{n,r}(f; \cdot)\|_\infty \leq K_r (\sigma/r\pi)^{n+1} \|f\|_\infty.$$

THEOREM 3. *If $f(x) = \int_{-r\pi}^{r\pi} e^{iux} d\alpha(u)$, where $\alpha(u)$ is a function of bounded variation in $[-r\pi, r\pi]$, then*

$$\lim_{n \rightarrow \infty} S_{n,r}(f; x) = f(x) + C 2^{r-1} \prod_{i=1}^r \sin \pi(x - x_i)$$

uniformly, where

$$C = \begin{cases} (-1)^r i \{ \alpha(r\pi) - \alpha(r\pi - 0) + \alpha(-r\pi) - \alpha(-r\pi + 0) \} & \text{if } n + r \text{ is even,} \\ (-1)^{r-1} \{ \alpha(r\pi) - \alpha(r\pi - 0) - \alpha(-r\pi) + \alpha(-r\pi + 0) \} & \text{if } n + r \text{ is odd.} \end{cases}$$

THEOREM 4. *If $f \in B_{r\pi}$ is almost periodic in the sense of Bohr, then*

$$\lim_{n \rightarrow \infty} S_{n,r}(f; x) = f(x) + C 2^r \prod_{i=1}^r \sin \pi(x - x_i)$$

uniformly, where

$$C = \begin{cases} (-1)^r \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x) \sin r\pi x dx & \text{if } n + r \text{ is even,} \\ (-1)^{r-1} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x) \cos r\pi x dx & \text{if } n + r \text{ is odd.} \end{cases}$$

In §2 we apply the results of our preceding paper [5] to the study of the sign structure of the kernel, from which the corresponding remainder formula is derived. The proof of Theorem 1 is given in §3. Theorem 1 is then used in §4 to derive convergence results.

2. The remainder formula. Take $1 \leq r \leq n+1$ and consider the poised, symmetric C.I.P. $(E, F, \mathbf{x})$ defined in §1. In this case the relation (1.5) and the conditions (a) and (b) imply that $0 < x_1 < \dots < x_r \leq 1$ with equality iff $n+r$ is even. We shall also write $\tilde{S}_{n,r}$ for $\mathcal{C}(F, \mathbf{x})$ and $\tilde{S}_{n,r}$ for $\mathcal{C}(E, \mathbf{x})$ which corresponds to the dual C.I.P. $(F, E, \mathbf{x})$.

Specialising the results of [5] to this special case it is easy to see that the corresponding null space

$$\mathcal{C}^0 \equiv \mathcal{C}^0(F, \mathbf{x}) := \{S \in \mathcal{C}(F, \mathbf{x}); S(\nu + x_i) = 0 \forall \nu \in \mathbf{Z}, E_{i0} = 1\}$$

has dimension d , where

$$d = \begin{cases} n-r & \text{if } n+r \text{ is even,} \\ n-r+1 & \text{if } n+r \text{ is odd,} \end{cases}$$

and is spanned by d eigensplines $S_j, j = 1, \dots, d$, satisfying the functional relation

$$(2.1) \quad S_j(x+1) = \lambda_j S_j(x) \quad \forall x \in \mathbf{R}.$$

The eigenvalues $\lambda_j, j = 1, \dots, d$, of the C.I.P. $(E, F, \mathbf{x})$ are real, distinct, of sign $(-1)^j$ and are precisely the eigenvalues of the matrix $C = (C_{\mu\nu})_{d \times d}$ where $S^\mu(1) = C_{\mu\nu} S^{(\nu)}(0), \mu, \nu = 0, \dots, d$.

Now for $i = 1, \dots, r$, we let L_i denote the unique element of $\mathcal{C}(F, \mathbf{x})$ of power growth (actually of exponential decay) satisfying

$$L_i(\nu + x_j) = \delta_{\nu 0} \delta_{ij}, \quad \forall \nu \in \mathbf{Z}, j = 1, \dots, r.$$

Then, for $f \in \mathfrak{F}_{n,r}$ of power growth, the unique function $S_{n,r}(f; \cdot) \in \mathcal{C}(F, \mathbf{x})$ of power growth that interpolates f for the C.I.P. $(E, F, \mathbf{x})$ is given by

$$(2.2) \quad S_{n,r}(f; x) = \sum_{i=1}^r \sum_{\nu=-\infty}^{\infty} f(\nu + x_i) L_i(x - \nu).$$

We let $\tilde{S}_{n,r}(f; \cdot) \in \tilde{S}_{n,r}$ denote the unique spline function of power growth that interpolates f for the dual C.I.P. $(F, E, \mathbf{x})$.

For $x, t \in \mathbf{R}$ we define

$$g_t(x) \equiv \tilde{g}_x(t) := (1/n!)(x-t)_+^n$$

and

$$(2.3) \quad K(x, t) \equiv K_t(x) \equiv \tilde{K}_x(t) := g_t(x) - S_{n,r}(g_t; x).$$

Clearly, from (2.1), we have

$$(2.4) \quad K(x+1, t+1) = K(x, t), \quad \forall x, t \in \mathbf{R},$$

and

$$(2.5) \quad K_t(\nu + x_i) = 0, \quad \forall \nu \in \mathbf{R} \text{ and } i = 1, \dots, r.$$

Now we see from (2.2) and (2.3) that, for $\rho = 0, \dots, n$,

$$\tilde{K}_x^{(\rho)}(t) = (-1)^\rho g_t^{(\rho)}(x) - S_{n,r}((-1)^\rho g_t^{(\rho)}; x).$$

But for $\nu \in \mathbf{Z}$ and $\rho = 0, \dots, r-1, g_\nu^{(\rho)} \in \mathcal{C}(F, \mathbf{x})$ and so

$$(2.6) \quad \tilde{K}_x^{(\rho)}(\nu) = 0, \quad \forall x \in \mathbf{R}.$$

Now we see from (2.2) that, for fixed x , $S_{n,r}(g_t; x)$ as a function of t lies in $\mathcal{C}(E, \mathbf{x})$ and has power growth. By (2.6) it interpolates $\tilde{g}_x$ for the C.I.P. $(F, E, \mathbf{x})$. Hence $S_{n,r}(g_t; x) = \tilde{S}_{n,r}(\tilde{g}_x; t)$, $\forall x, t \in \mathbf{R}$, and so

$$(2.7) \quad K(x, t) = \tilde{g}_x(t) - \tilde{S}_{n,r}(\tilde{g}_x; t).$$

LEMMA 2.1. *For $1 < r < n$, there is a constant $\beta > 0$ such that the following hold for $\rho = 0, \dots, n$.*

(a) *For $t \in \mathbf{R}$, $K_t^{(\rho)}(x) = O(e^{-\beta|x|})$ as $x \rightarrow \pm\infty$ and the only zeros of K_t ($t \notin \mathbf{Z}$) are simple zeros at $\nu + x_i$, $\nu \in \mathbf{Z}$, $i = 1, \dots, r$.*

(b) *For $x \in \mathbf{R}$, $\tilde{K}_x^{(\rho)}(t) = O(e^{-\beta|t|})$ as $t \rightarrow \pm\infty$ and the only zeros of $\tilde{K}_x$ ($x \notin \mathbf{Z} + x_i$) are isolated zeros of multiplicity r at the integers.*

PROOF. We shall prove only (a) as (b) follows similarly by duality, and only for even $n + r$ as the result for odd $n + r$ follows similarly. By (2.4) and (2.6) we may suppose $0 < t < 1$.

Now, for $n + r$ even the C.I.P. $(E, F, \mathbf{x})$ has $n - r$ distinct eigenvalues $\lambda_1, \dots, \lambda_{n-r}$ of sign $(-1)^r$ with $|\lambda_1| > |\lambda_2| > \dots > |\lambda_{n-r}| > 0$ and $\lambda_{n-r-i+1} = \lambda_i^{-1}$, $i = 1, \dots, n - r$.

We let $S_1, \dots, S_{n-r}$ denote the corresponding eigensplines, which span $\mathcal{C}^0$. The eigenvalues $\lambda_1, \dots, \lambda_{n-r}$ are precisely the eigenvalues of the matrix C where $(-1)^r C$ is an oscillation matrix with corresponding eigenvectors $(S'_i(0), \dots, S_i^{(n-r)}(0))$, $i = 1, \dots, n - r$. So by a theorem of Gantmacher and Krein we have

$$S(S'_i(0), \dots, S_i^{(n-r)}(0)) = i - 1, \quad i = 1, \dots, n - r$$

(see also Micchelli [7]).

Now $K_t \in C^{n-r}(\mathbf{R})$ and for $\nu = 1, 2, \dots$, $K_t|_{(\nu, \nu+1)} \in \pi_n$. Thus, by (2.5), $K_t|_{[1, \infty)}$ can be extended to an element of $\mathcal{C}^0$. Since K_t is of power growth, we therefore have

$$K_t(x) = \sum_{i \geq p} c_i S_i(x), \quad \forall x \geq 1,$$

where $c_p \neq 0$ and $p > \frac{1}{2}(n - r)$. Similarly $K_t(x) = \sum_{i \leq q} c_i S_i(x)$, $\forall x \leq 0$, where $c_q \neq 0$ and $q < \frac{1}{2}(n - r)$.

The first part of (a) follows.

Now for $\rho = 0, \dots, n - r$ and $\nu = 1, 2, \dots$,

$$K_t^{(\rho)}(\nu) = \sum_{i \geq p} c_i \lambda_i^\nu S_i^{(\rho)}(0) \quad \text{and} \quad K_t^{(\rho)}(-\nu) = \sum_{i \leq q} c_i \lambda_i^{-\nu} S_i^{(\rho)}(0).$$

So $K_t^{(\rho)}(\nu) = c_p \lambda_p^\nu S_p^{(\rho)}(0) + O(\lambda_p^\nu)$ as $\nu \rightarrow \infty$ and $K_t^{(\rho)}(-\nu) = c_q \lambda_q^{-\nu} S_q^{(\rho)}(0) + O(\lambda_q^{-\nu})$ as $\nu \rightarrow \infty$. Thus for large enough N ,

$$(2.8) \quad S(K'_t(N), \dots, K_t^{(n-r)}(N)) = S(S'_p(0), \dots, S_p^{(n-r)}(0)) = p - 1$$

and

$$(2.9) \quad S(K'_t(-N), \dots, K_t^{(n-r)}(-N)) = S(S'_q(0), \dots, S_q^{(n-r)}(0)) = q - 1.$$

We now show that K_t is oscillating in $(-N, N)$. For suppose $K_t = 0$ on some interval $(a, b) \in (-N, N)$. Since $K_t \in C^{n-r}(\mathbf{R})$ and is a piecewise polynomial with

knots at $\mathbf{Z} \cup \{t\}$, $a, b \in \mathbf{Z} \cup \{t\}$. Furthermore, either $a < t$ or $b > t$. Suppose $a < t$. Then $K_t^{(\rho)}(a) = 0$, $\rho = 0, \dots, n-r$, and $K_t(a-1+x_i) = 0$, $i = 0, \dots, r-1$, implies that $K_t = 0$ on $(a-1, a)$. Hence $K_t^{(\rho)}(a-1) = 0$, $\rho = 0, \dots, n-r$. By induction we have $K_t^{(\rho)}(-N) = 0$, $\rho = 0, \dots, n-r$, which contradicts (2.9). Similarly $b > t$ leads to $K_t^{(\rho)}(N) = 0$, $\rho = 0, \dots, n-r$, which contradicts (2.8).

Now K_t has exact degree n in $(0, 1)$ and so we may apply Theorem 2.1 of [5] to $g := K_t|_{(-N, N)}$ to give

$$\begin{aligned} Z(g) &\leq (2N-1)r+1 + S^-(g(-N^+), \dots, g^{(n)}(-N^+)) \\ &\quad - S^+(g(N^-), \dots, g^{(n)}(N^-)) \\ &\leq (2N-1)r+1 + (q-1+r) - p \quad (\text{since } g(-N^+) = g(N^-) = 0) \\ &= 2Nr + q - p \leq 2Nr - 1. \end{aligned}$$

But K_t has $2Nr-1$ zeros in $(-N, N)$ at points as in (2.5) and hence these are the only zeros of K_t in $(-N, N)$. Since N can be arbitrarily large, (a) follows. $\square$

We must also consider the special cases $r = n$ or $n+1$, for which it is easy to see the following. If $t \in (\nu, \nu+1)$, then K_t vanishes outside $[\nu, \nu+1]$ and vanishes in $(\nu, \nu+1)$ only at $\nu+x_i$, $i = 1, \dots, r$. Similarly if $x \in (\nu, \nu+1)$, then $\tilde{K}_x$ vanishes outside $[\nu, \nu+1]$ and vanishes nowhere in $(\nu, \nu+1)$.

THEOREM 2.1. *If $f \in \mathfrak{G}_{n,r}$ and $f^{(n+1-r)}$ is of power growth, then*

$$(2.10) \quad f(x) - S_{n,r}(f; x) = \int_{-\infty}^{\infty} K(x, t) f^{(n+1)}(t) dt.$$

PROOF. By (2.4) we may assume $0 < x \leq 1$. We may also assume $x \neq x_i$, $i = 1, \dots, r$, since otherwise (2.10) is trivially satisfied.

Now integrating by parts and applying (2.6) gives, for any $\nu \in \mathbf{Z}$,

$$\int_{\nu}^{\nu+1} K(x, t) f^{(n+1)}(t) dt = (-1)^r \int_{\nu}^{\nu+1} \tilde{K}_x^{(r)}(t) f^{(n+1-r)}(t) dt.$$

So $\int_{-\infty}^{\infty} K(x, t) f^{(n+1)}(t) dt = (-1)^r \int_{-\infty}^{\infty} \tilde{K}_x^{(r)}(t) f^{(n+1-r)}(t) dt$ which converges since $f^{(n+1-r)}$ is of power growth and $\tilde{K}_x^{(r)}$ decays exponentially. Also $f^{(\rho)}$ is of power growth for $0 \leq \rho \leq n+1-r$ and so we may integrate by parts to give

$$(-1)^r \int_{-\infty}^{\infty} \tilde{K}_x^{(r)}(t) f^{(n+1-r)}(t) dt = (-1)^n \int_{-\infty}^{\infty} \tilde{K}_x^{(n)}(t) f'(t) dt.$$

Now $(-1)^n \tilde{K}_x^{(n)}(t) = (x-t)_+^0 - \sum_{i=1}^r \sum_{\nu=-\infty}^{\infty} (\nu+x_i-t)_+^0 L_i(x-\nu)$ and so

$$\begin{aligned} (-1)^n \int_{-\infty}^{\infty} \tilde{K}_x^{(n)}(t) f'(t) dt &= f(x) - \sum_{i=1}^r \sum_{\nu=-\infty}^{\infty} f(\nu+x_i) L_i(x-\nu) \\ &= f(x) - S_{n,r}(f; x). \quad \square \end{aligned}$$

3. Proof of Theorem 1. We see immediately from Theorem 2.1 that for $f \in \mathfrak{G}_{n,r}$ with $f^{(n+1-r)}$ of power growth and $\|f^{(n+1)}\|_{\infty} < \infty$

$$\|f - S_{n,r}(f;)\|_{\infty} \leq \sup_{x \in \mathbf{R}} \left\{ \int_{-\infty}^{\infty} |K(x, t)| dt \right\} \|f^{(n+1)}\|_{\infty}.$$

Now there is an eigenspline $E_{n+1,r} \in \mathfrak{S}_{n+1,r}$ with eigenvalue $(-1)^r$ which vanishes at $\nu + x_i$, $\forall \nu \in \mathbf{Z}$ and $i = 1, \dots, r$. We assume it is normalised so that $|E_{n+1,r}^{(n+1)}| = 1$.

Putting $f = E_{n+1,r}$ in (2.10), we see from Lemma 2.1(b) that

$$|E_{n+1,r}(x)| = \int_{-\infty}^{\infty} |K(x, t)| dt, \quad \forall x \in \mathbf{R},$$

so that the constant K in (1.6) is given by

$$(3.1) \quad K = \|E_{n+1,r}\|_{\infty}.$$

Following the definitions of Cavaretta [1] and Goodman [4], we let $\mathfrak{S}_{n+1,r} \in \mathfrak{S}_{n+1,r}$ denote the Euler-Chebyshev spline, normalised so that $|\mathfrak{S}_{n+1,r}^{(n+1)}| = 1$. Now the zeros of $\mathfrak{S}_{n+1,r}$ are points $\nu + \beta_i$, $\nu \in \mathbf{Z}$, where β_i , $i = 1, \dots, r$, are symmetric about $x = \frac{1}{2}$, and $0 < \beta_1 < \dots < \beta_r \leq 1$, with equality iff $n + r$ is even. Furthermore

$$\mathfrak{S}_{n+1,r}(x+1) = (-1)^r \mathfrak{S}_{n+1,r}(x), \quad \forall x \in \mathbf{R},$$

and $f = \mathfrak{S}_{n+1,r}$ minimises $\|f\|_{\infty}$ over all $f \in \mathfrak{S}_{n+1,r}$ with $\|f^{(n+1)}\|_{\infty} = 1$. The result follows. $\square$

4. Convergence of $S_{n,r}(f; \cdot)$. Henceforth we shall examine the behaviour of $S_{n,r}(f, x)$ as $n \rightarrow \infty$. Analogous problems were studied in [6, 9 and 10]. We first derive an estimate for $\|E_{n+1,r}\|_{\infty}$.

LEMMA 4.1. *For fixed x and $r \geq 1$, $\exists K_r$ such that*

$$(4.1) \quad \|E_{n+1,r}\|_{\infty} \leq K_r / (r\pi)^{n+1}, \quad \forall n \geq r-1.$$

PROOF. First, assume that r is odd. For $x \in [0, 1]$, let

$$E_{n+1,r}(x) = a_0 E_{n+1}(x) + a_1 E_n(x) + \dots + a_n E_1(x) + a_{n+1} E_0(x),$$

where $E_k(x)$, $k = 0, 1, \dots, n+1$, are the Euler polynomials. The conditions $E_{n+1,r}^{(\rho)}(1) = -E_{n+1,r}^{(\rho)}(0)$ for $\rho = 0, 1, \dots, n-r+1$ imply that $a_{n-k+1} = 0 \quad \forall k = 0, 1, \dots, n-r+1$. Hence for $x \in [0, 1]$,

$$E_{n+1,r}(x) = a_0 E_{n+1}(x) + a_1 E_n(x) + \dots + a_{r-1} E_{n-r+2}(x).$$

Now $E_{n+1,r}(x_i) = 0$, $i = 1, 2, \dots, r$, gives a homogeneous system of equations $a_0 E_{n+1}(x_i) + a_1 E_n(x_i) + \dots + a_{r-1} E_{n-r+2}(x_i) = 0$, $i = 1, 2, \dots, r$, whose determinant must be zero. Hence we can write

$$E_{n+1,r}(x) = \frac{\det(E_{n-m+2}(\beta_l))_{l,m=1}^r}{\det(E_{n-m+2}(x_l))_{l,m=2}^r} \quad \forall x \in [0, 1]$$

where

$$\beta_l = \begin{cases} x & \text{if } l = 1, \\ x_l & \text{if } l \neq 1. \end{cases}$$

Using the Fourier expansions of the Euler polynomials $E_k(x)$ we obtain

$$|E_{n+1,r}(x)| = \frac{2}{\pi^{n+2}} \left| \frac{\det \left(\sum_{l=1}^r e^{2k_m \pi i \beta_l} / (2k+1)^{n-m+3} \right)_{l,m=1}^r}{\det \left(\sum_{l=2}^r e^{2k_m \pi i x_l} / (2k+1)^{n-m+3} \right)_{l,m=2}^r} \right|$$

$$= \left(\frac{2^r}{\pi^{n+2}} \right) \left| \frac{\sum_{k_1, k_2, \dots, k_r} V(k_1, k_2, \dots, k_r) \prod_{j=1}^r e^{2k_j \pi i \beta_l} / (2k_j + 1)^{n+1}}{\sum_{k_2, k_3, \dots, k_r} V(k_2, k_3, \dots, k_r) \prod_{j=2}^r e^{2k_j \pi i x_l} / (2k_j + 1)^{n+1}} \right|$$

where $V(a_1, a_2, \dots, a_r)$ denotes the Vandermonde determinant $\det(a_m^{l-1})_{l,m=1}^r = \prod_{1 \leq j < k \leq r} (a_k - a_j)$. A straightforward computation gives

$$(4.2) \quad \left(\frac{2^r}{\pi^{n+2}} \right) \left| \frac{\sum_{k_1 < \dots < k_r} V(k_1, k_2, \dots, k_r) \det(e^{2k_m \pi i \beta_l})_{l,m=1}^r \sum_{j=1}^r (2k_j + 1)^{-(n+2)}}{\sum_{k_2 < \dots < k_r} V(k_2, k_3, \dots, k_r) \det(e^{2k_m \pi i x_l})_{l,m=2}^r \sum_{j=2}^r (2k_j + 1)^{-(n+1)}} \right|.$$

The dominant term in the numerator of (4.2) is

$$(4.3) \quad \prod_{j=1}^r (2j - r)^{-(n+2)} \left\{ V \left(-\frac{(r-1)}{2}, -\frac{(r-3)}{2}, \dots, \frac{r-1}{2} \right) (e^{(2m-r-1)\pi i \beta_l})_{l,m=1}^r \right. \\ \left. - V \left(-\frac{(r+1)}{2}, -\frac{(r-1)}{2}, \dots, \frac{r-3}{2} \right) (e^{(2m-r-3)\pi i \beta_l})_{l,m=1}^r \right\}.$$

The dominant term in the denominator is

$$(4.4) \quad \prod_{j=2}^r (2j - r - 2)^{-(n+1)} V \left(-\frac{(r-1)}{2}, -\frac{(r-3)}{2}, \dots, \frac{(r-3)}{2} \right) \\ \times \det(e^{(2m-r-3)\pi i x_l})_{l,m=2}^r.$$

It follows from (4.2), (4.3) and (4.4) that $\forall x \in [0, 1]$

$$|E_{n+1,r}(x)| = \left(\frac{2^r}{\pi^{n+2}} \right) \left| \frac{\prod_{j=2}^r (2j - r - 2)^{n+1}}{\prod_{j=1}^r (2j - r)^{n+2}} \right| O(1),$$

and (4.1) follows for odd r .

If r is even, a similar argument shows that for $x \in [0, 1]$ the eigensplines $E_{n+1,r}(x)$ may be expressed in terms of Bernoulli polynomials $B_k(x)$ as follows:

$$E_{n+1,r}(x) = \frac{\det^*(B_{n-m+3}(\beta_l))_{l=1; m=2}^r}{\det^*(B_{n-m+3}(x_l))_{l=2; m=3}^r},$$

where $\det^*$ means that all the entries in the last row of the determinant are 1. Expanding each determinant along the last row and applying a similar method to each term gives the inequality (4.1) for r even. $\square$

Now recall the definition of B_σ in §1. We shall need Bernstein's theorem that if $f \in B_\sigma$ then, for each integer n , $f^{(n)} \in B_\sigma$ and

$$(4.5) \quad \|f^{(n)}\|_\infty \leq \sigma^n \|f\|_\infty.$$

From (1.6), (3.1), (4.1) and (4.5), we can immediately deduce Theorem 2.

COROLLARY 4.1. *If $f \in B_\sigma$ and $\sigma < r\pi$, then*

$$\lim_{n \rightarrow \infty} S_{n,r}(f; x) = f(x) \quad \text{uniformly on } \mathbf{R}.$$

COROLLARY 4.2. *If $f \in B_{r\pi}$, then for all $n \geq r - 1$*

$$\|f - S_{n,r}(f)\|_\infty \leq K_r \|f\|_\infty.$$

We now follow a similar approach to that of Schoenberg [10] in proving Theorem 3. First we introduce the class $B_{r\pi}^*$ of functions which are uniform limits on $\mathbf{R}$ of functions belonging to B_p for $p < r\pi$, i.e. $f \in B_{r\pi}^*$ if and only if $\exists f_j \in B_{p_j}$, $p_j < r\pi$, $j = 1, 2, 3, \dots$, such that $\|f - f_j\|_\infty \rightarrow 0$ as $j \rightarrow \infty$.

LEMMA 4.2. *If $f \in B_{r\pi}^*$ then*

$$\lim_{n \rightarrow \infty} S_{n,r}(f; x) = f(x) \quad \text{uniformly}.$$

PROOF. Suppose $f_j \in B_{p_j}$, $p_j < r\pi$, $j = 1, 2, 3, \dots$, and $\|f - f_j\|_\infty \rightarrow 0$ as $j \rightarrow \infty$. Then $f - S_{n,r}(f; \cdot) = f - f_j - S_{n,r}(f - f_j; \cdot) + f_j - S_{n,r}(f_j; \cdot)$ and using Corollaries 4.1 and 4.2, the result follows as in [10]. $\square$

Before we prove Theorems 3 and 4 we first study the behaviour of the spline functions $S_{n,r}(\cos r\pi x)$ and $S_{n,r}(\sin r\pi x)$ that interpolate $\cos r\pi x$ and $\sin r\pi x$ respectively at $\nu + x_i$, $i = 1, 2, \dots, r$, $\nu \in \mathbf{Z}$. In order to simplify writing, we define $\alpha_1, \dots, \alpha_r$ by

$$\begin{aligned} \alpha_i &= x_i, & i &= 1, \dots, r, \text{ if } n + r \text{ is odd,} \\ \alpha_1 &= 0 \quad \text{and} \quad \alpha_i &= x_{i-1}, & i &= 1, \dots, r - 1, \text{ if } n + r \text{ is even.} \end{aligned}$$

We first introduce the exponential Euler splines

$$(4.6) \quad S_{n,r}(x; u) = \sum_{s=1}^r e^{iux} \Omega_s(x, u) \quad \forall x \in \mathbf{R},$$

where

$$\Omega_s(x, u) = \frac{\det(\sum_{k=-\infty}^{\infty} e^{2k\pi i \beta_l} / (u + 2k\pi)^{n-m+2})_{l,m=1}^r}{\det(\sum_{k=-\infty}^{\infty} e^{2k\pi i \alpha_l} / (u + 2k\pi)^{n-m+2})_{l,m=1}^r},$$

$(r-2)\pi < u \leq r\pi$, and

$$\beta_l = \begin{cases} x & \text{if } l = s, \\ \alpha_l & \text{if } l \neq s. \end{cases}$$

Clearly $S_{n,r}(\nu + \alpha_i; u) = e^{iu(\nu + \alpha_i)} \quad \forall i = 1, 2, \dots, r, \quad \nu \in \mathbf{Z}$, and $S_{n,r}(x, r\pi) = S_{n,r}(\cos r\pi x) + iS_{n,r}(\sin r\pi x)$. Therefore we are interested in the limit of $S_{n,r}(x; r\pi)$ as $n \rightarrow \infty$. First we prove

LEMMA 4.3. For $s = 1, 2, \dots, r$,

$$(4.7) \quad \lim_{n \rightarrow \infty} \Omega_s(x, r\pi) = e^{\pi i(\alpha_s - x)} \cos \pi(\alpha_s - x) \frac{V(e^{-2\pi i\beta_1}, e^{-2\pi i\beta_2}, \dots, e^{-2\pi i\beta_r})}{V(e^{-2\pi i\alpha_1}, e^{-2\pi i\alpha_2}, \dots, e^{-2\pi i\alpha_r})}$$

uniformly on $\mathbf{R}$.

PROOF. Assume $(r-1)\pi < u < r\pi$. A straightforward computation shows that

$$\begin{aligned} \Omega_s(x, u) &= \frac{\sum_{k_1, k_2, \dots, k_r} V(k_1, k_2, \dots, k_r) \prod_{j=1}^r e^{2k_j \pi i \beta_j} / (u + 2k_j \pi)^{n+1}}{\sum_{k_1, k_2, \dots, k_r} V(k_1, k_2, \dots, k_r) \prod_{j=1}^r e^{2k_j \pi i \alpha_j} / (u + 2k_j \pi)^{n+1}} \\ &= \frac{\sum_{k_1 < \dots < k_r} V(k_1, k_2, \dots, k_r) \det(e^{2k_m \pi i \beta_l})_{l,m=1}^r \prod_{j=1}^r (u + 2k_j \pi)^{-n-1}}{\sum_{k_1 < \dots < k_r} V(k_1, k_2, \dots, k_r) \det(e^{2k_m \pi i \alpha_l})_{l,m=1}^r \prod_{j=1}^r (u + 2k_j \pi)^{-n-1}} \\ &= \frac{\det(e^{-2(m-1)\pi i \beta_l})_{l,m=1}^r + \left(\frac{u}{u-2r\pi}\right)^{n+1} \det(e^{-2m\pi i \beta_l})_{l,m=1}^r + O\left(\left|\frac{u-2\pi}{u-2r\pi}\right|^{n+1}\right)}{\det(e^{-2(m-1)\pi i \alpha_l})_{l,m=1}^r + \left(\frac{u}{u-2r\pi}\right)^{n+1} \det(e^{-2m\pi i \alpha_l})_{l,m=1}^r + O\left(\left|\frac{u-2\pi}{u-2r\pi}\right|^{n+1}\right)}. \end{aligned}$$

Taking the limit as $u \rightarrow r\pi$, after some simplification, we obtain

$$(4.8) \quad \Omega_s(x, u) = \frac{V(e^{-2\pi i\beta_1}, \dots, e^{-2\pi i\beta_r}) \left(1 + (-1)^{n+1} e^{-2\pi i \sum_{l=1}^r \beta_l}\right) + O((r-2/r)^{n+1})}{V(e^{-2\pi i\alpha_1}, \dots, e^{-2\pi i\alpha_r}) \left(1 + (-1)^{n+1} e^{-2\pi i \sum_{l=1}^r \alpha_l}\right) + O((r-2/r)^{n+1})}.$$

Now suppose n even. If r is even, $\alpha_1 = 0$ and

$$(4.9) \quad \sum_{l=1}^r \alpha_l = \frac{r-2}{2} + \frac{1}{2}.$$

If r is odd, $\alpha_1 > 0$ and

$$(4.10) \quad \sum_{l=1}^r \alpha_l = \frac{r-1}{2} + \frac{1}{2}.$$

The result (4.7) then follows from (4.8), (4.9) and (4.10). For odd n ,

$$\sum_{l=1}^r \alpha_l = \begin{cases} (r-1)/2 & \text{if } r \text{ is odd,} \\ r/2 & \text{if } r \text{ is even,} \end{cases}$$

and (4.7) follows similarly. $\square$

LEMMA 4.4. If $\alpha_1 = 0$, the following limits hold uniformly:

$$(4.11) \quad \lim_{n \rightarrow \infty} S_{n,r}(\cos r\pi x) = \cos r\pi x,$$

$$(4.12) \quad \lim_{n \rightarrow \infty} S_{n,r}(\sin r\pi x) = \sin r\pi x + (-1)^r 2^{r-1} \prod_{i=1}^r \sin \pi(x - \alpha_i).$$

If $\alpha_1 > 0$, the following limits hold uniformly:

$$(4.13) \quad \lim_{n \rightarrow \infty} S_{n,r}(\cos r\pi x) = \cos r\pi x + (-1)^{r-1} 2^{r-1} \prod_{i=1}^r \sin \pi(x - \alpha_i),$$

$$(4.14) \quad \lim_{n \rightarrow \infty} S_{n,r}(\sin r\pi x) = \sin r\pi x.$$

PROOF. First we write

$$\begin{aligned} \frac{V(e^{-2\pi i\beta_1}, e^{-2\pi i\beta_2}, \dots, e^{-2\pi i\beta_r})}{V(e^{-2\pi i\alpha_1}, e^{-2\pi i\alpha_2}, \dots, e^{-2\pi i\alpha_r})} &= \frac{\prod_{1 \leq l < k \leq r} (e^{-2\pi i\beta_k} - e^{-2\pi i\beta_l})}{\prod_{1 \leq l < k \leq r} (e^{-2\pi i\alpha_k} - e^{-2\pi i\alpha_l})} \\ &= e^{(r-1)\pi i(\alpha_s - x)} \frac{\prod_{1 \leq l < k \leq r} \sin \pi(\beta_k - \beta_l)}{\prod_{1 \leq l < k \leq r} \sin \pi(\alpha_k - \alpha_l)} \\ &= e^{(r-1)\pi i(\alpha_s - x)} \frac{\prod'_{k=1}^r \sin \pi(x - \alpha_k)}{\prod'_{k=1}^r \sin \pi(\alpha_s - \alpha_k)}, \end{aligned}$$

where $\prod'_{k=1}^r$ indicates that the factor involving $k = s$ is omitted. Hence it follows from (4.6) and (4.7) that

$$\begin{aligned} \lim_{n \rightarrow \infty} S_{n,r}(x, r\pi) &= \sum_{s=1}^r e^{r\pi i\alpha_s} \cos \pi(x - \alpha_s) \frac{\prod'_{k=1}^r \sin \pi(x - \alpha_k)}{\prod'_{k=1}^r \sin \pi(\alpha_s - \alpha_k)} \\ &\equiv \phi(x) + i\psi(x), \end{aligned}$$

where

$$\phi(x) = \sum_{s=1}^r \cos r\pi\alpha_s \cos \pi(x - \alpha_s) \frac{\prod'_{k=1}^r \sin \pi(x - \alpha_k)}{\prod'_{k=1}^r \sin \pi(\alpha_s - \alpha_k)},$$

and

$$\psi(x) = \sum_{s=1}^r \sin r\pi\alpha_s \cos \pi(x - \alpha_s) \frac{\prod'_{k=1}^r \sin \pi(x - \alpha_k)}{\prod'_{k=1}^r \sin \pi(\alpha_s - \alpha_k)}.$$

Clearly $\lim_{n \rightarrow \infty} S_{n,r}(\cos r\pi x) = \phi(x)$, $\lim_{n \rightarrow \infty} S_{n,r}(\sin r\pi x) = \psi(x)$.

Next, we want to simplify $\phi(x)$ and $\psi(x)$. First we consider r even. The Gauss trigonometric interpolation formula gives

$$(4.15) \quad \phi(x) = \cos r\pi x + \lambda \prod'_{k=1}^r \sin \pi(x - \alpha_k).$$

To determine λ , we write

$$(4.16) \quad \cos \pi(\alpha_s - x) \prod'_{k=1}^r \sin \pi(x - \alpha_k) = \frac{1}{2(2i)^{r-1}} (e^{\pi i(x - \alpha_s)} + e^{-\pi i(x - \alpha_s)}) \\ \cdot \left(\exp \left[\pi i \left((r-1)x - \sum'_{k=1}^r \alpha_k \right) \right] + \cdots + (-1)^{r-1} \exp \left[-\pi i \left((r-1)x - \sum'_{k=1}^r \alpha_k \right) \right] \right)$$

and

$$(4.17) \quad \prod'_{k=1}^r \sin \pi(x - \alpha_k) = \frac{1}{(2i)^r} \left(\exp \left[\pi i \left(rx - \sum_{k=1}^r \alpha_k \right) \right] + \cdots \right. \\ \left. + (-1)^r \exp \left[-\pi i \left(rx - \sum_{k=1}^r \alpha_k \right) \right] \right).$$

Equating the highest order terms in (4.15), it follows from (4.15)–(4.17) that

$$\frac{1}{2(2i)^{r-1}} \sum_{s=1}^r A_s \left\{ \cos \left(\pi \left(rx - \sum_{k=1}^r \alpha_k \right) \right) + i \sin \left(\pi \left(rx - \sum_{k=1}^r \alpha_k \right) \right) \right. \\ \left. + (-1)^{r-1} \left[\cos \left(\pi \left(rx - \sum_{k=1}^r \alpha_k \right) \right) - i \sin \left(\pi \left(rx - \sum_{k=1}^r \alpha_k \right) \right) \right] \right\} \\ = \cos r\pi x + \lambda \left\{ \cos \pi \left(rx - \sum_{k=1}^r \alpha_k \right) + i \sin \pi \left(rx - \sum_{k=1}^r \alpha_k \right) \right. \\ \left. + (-1)^r \left[\cos \pi \left(rx - \sum_{k=1}^r \alpha_k \right) - i \sin \pi \left(rx - \sum_{k=1}^r \alpha_k \right) \right] \right\},$$

where $A_s = \cos \pi r \alpha_s / \prod'_{k=1}^r \sin \pi(\alpha_s - \alpha_k)$.

Since r is even it follows that

$$(4.18) \quad \frac{-2}{(2i)^r} \sum_{s=1}^r A_s \sin \pi \left(rx - \sum_{k=1}^r \alpha_k \right) = \cos r\pi x + \frac{2\lambda \cos \pi \left(rx - \sum_{k=1}^r \alpha_k \right)}{(2i)^r}.$$

Now if $\alpha_1 = 0$, then $\sum_{k=1}^r \alpha_k = r/2 - \frac{1}{2}$, so that (4.18) becomes

$$\frac{(-1)^{(r+2)/2}}{(2i)^r} \cos r\pi x \left(\sum_{s=1}^r A_s \right) = \cos r\pi x + \frac{(-1)^{(r+2)/2} 2\lambda \sin r\pi x}{(2i)^r}.$$

Hence $\lambda = 0$. This proves (4.11) for r even.

If $\alpha_1 \neq 0$, then $\sum_{k=1}^r \alpha_k = r/2$, so that (4.18) becomes

$$\frac{-2}{2^r} \sin r\pi x \left(\sum_{s=1}^r A_s \right) = \cos r\pi x + \frac{2\lambda \cos r\pi x}{2^r}.$$

Hence $\lambda = -2^{r-1}$. This proves (4.13) for r even. The proof of (4.12) and (4.14) for r even are the same.

Next we consider r odd. If $\alpha_1 \neq 0$, we let $\alpha_0 = 0$ and write

$$(4.19) \quad \phi(x) = \frac{1}{\sin \pi x} \sum_{s=0}^r \cos r\pi \alpha_s \cos \pi(x - \alpha_s) \frac{\prod_{k=0}^{r-1} \sin \pi(x - \alpha_k)}{\prod_{k=0}^{r-1} \sin \pi(\alpha_s - \alpha_k)}.$$

The Gauss interpolation formula again gives

$$\begin{aligned} & \sum_{s=0}^r \cos r\pi \alpha_s \sin \pi \alpha_s \cos \pi(x - \alpha_s) \frac{\prod_{k=0}^{r-1} \sin \pi(x - \alpha_k)}{\prod_{k=0}^{r-1} \sin(\alpha_s - \alpha_k)} \\ &= \cos r\pi x \sin \pi x + \lambda \prod_{k=0}^{r-1} \sin \pi(x - \alpha_k). \end{aligned}$$

A similar calculation gives $\lambda = 2^{r-1}$, so that (4.19) gives $\phi(x) = \cos r\pi x + 2^{r-1} \prod_{k=1}^r \sin \pi(x - \alpha_k)$. This proves (4.13) for r odd. The proof of (4.14) is similar.

If $\alpha_1 = 0$, we let $\alpha_{k+1} = \frac{1}{2}$ and write

$$(4.20) \quad \phi(x) = \frac{1}{\cos \pi x} \sum_{s=1}^{r+1} \cos r\pi \alpha_s \cos \pi \alpha_s \cos \pi(x - \alpha_s) \frac{\prod_{k=1}^{r+1} \sin \pi(x - \alpha_k)}{\prod_{k=1}^{r+1} \sin \pi(\alpha_s - \alpha_k)},$$

and (4.11) and (4.12) for odd r are proved similarly. $\square$

PROOF OF THEOREM 3. Let

$$\alpha_0(u) = \begin{cases} \alpha(-r\pi + 0) & \text{if } u = -r\pi, \\ \alpha(u) & \text{if } -r\pi < u < r\pi, \\ \alpha(r\pi - 0) & \text{if } u = r\pi. \end{cases}$$

Then $\alpha_0(u)$ has no jumps at $\pm r\pi$. Define

$$f_0(x) = \int_{-r\pi}^{r\pi} e^{iux} d\alpha_0(u) \quad \forall x \in \mathbf{R}.$$

Setting $A_1 = \alpha(-r\pi + 0) - \alpha(-r\pi)$, $A_2 = \alpha(r\pi) - \alpha(r\pi - 0)$, we can write $f(x) = f_0(x) + A_1 e^{-r\pi i x} + A_2 e^{r\pi i x}$, and setting $A = A_1 + A_2$, $B = i(A_2 - A_1)$ we obtain

$$(4.21) \quad f(x) = f_0(x) + A \cos r\pi x + B \sin r\pi x \quad \forall x \in \mathbf{R}.$$

Now $f_0 \in B_{r\pi}^*$ since f_0 is the uniform limit of the sequence $\{f_j\}$, $f_j \in B_{p_j}$, defined by $f_j(x) = \int_{-p_j}^{p_j} e^{iux} d\alpha_0(u)$, with $0 < p_j < r\pi$, $p_j \rightarrow r\pi$ as $j \rightarrow \infty$. By Lemma 4.2 we conclude that $\lim_{n \rightarrow \infty} S_{n,r}(f_0; x) = f_0(x)$ uniformly on $\mathbf{R}$. The theorem now follows from (4.21) and Lemma 4.4. $\square$

Finally we consider the class $\mathcal{Q}^{\mathfrak{P}}$ of almost periodic functions in the sense of Bohr. To every $f \in \mathcal{Q}^{\mathfrak{P}}$ corresponds a Fourier series

$$f(x) \sim \sum_{\nu=1}^{\infty} A_{\nu} e^{i\lambda_{\nu} x},$$

where λ_{ν} are real numbers, called the Fourier exponents of f . Also for $\sigma \geq 0$,

$$\mathcal{Q}^{\mathfrak{P}} \cap B_{\sigma} = \{f: f \in \mathcal{Q}^{\mathfrak{P}}, -\sigma \leq \lambda_{\nu} \leq \sigma\}.$$

PROOF OF THEOREM 4. Suppose $f \in \mathcal{Q}^{\mathfrak{P}} \cap B_{r\pi}$. Then its Fourier exponents λ_{ν} , $\nu = 1, 2, 3, \dots$, satisfy $-r\pi \leq \lambda_{\nu} \leq r\pi$.

Without loss of generality we may assume that $\lambda_1 = -r\pi$, $\lambda_2 = r\pi$ with the understanding that $A_1 = 0$ if the exponent $-r\pi$ is absent, and similarly that $A_2 = 0$ if exponent $r\pi$ is absent.

Let

$$A_1 e^{-r\pi i x} + A_2 e^{r\pi i x} = A \cos r\pi x + B \sin r\pi x,$$

where $A = A_2 + A_1$, $B = i(A_2 - A_1)$. It follows that the function

$$(4.22) \quad g(x) = f(x) - A \cos r\pi x - B \sin r\pi x$$

has Fourier series $g(x) \sim \sum_{\nu=3}^{\infty} A_{\nu} e^{i\lambda_{\nu} x}$ where $-r\pi < \lambda_{\nu} < r\pi \quad \forall \lambda = 3, 4, 5, \dots$. A similar argument as in [10] shows that $g \in B_{r\pi}^*$. It follows from Lemma 4.2 that

$$(4.23) \quad S_{n,r}(g; x) \rightarrow g(x) \quad \text{uniformly on } \mathbf{R}.$$

The theorem then follows from (4.22), (4.23) and Lemma 4.4. $\square$

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